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Non-critical string pentagon equations and their solutions

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Abstract

We derive pentagon-type relations for the three-point boundary tachyon correlation functions in the non-critical open string theory with generic $c_{\text{matter}} < 1$ and study their solutions in the case of Fateev–Zamolodchikov–Zamolodchikov branes. A new general formula for the Liouville boundary three-point factor corresponding to degenerate matter is derived.

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1. Introduction

The associativity of the operator product expansion (OPE) of the boundary fields implies an equation [1] for the boundary three-point functions. It can be rewritten [2] as a pentagon-type relation for the boundary OPE coefficients, similar to the pentagon relation for the fusing matrix, the quantum $6j$ symbols. The two equations are identified in the rational case [3] as part of the Big Pentagon relations of a weak C^* -Hopf algebra [4, 5] interpreted as the quantum symmetry of the given BCFT. The boundary field OPE coefficients play the role of the quantum $3j$ symbols of this algebra. In the CFT described by diagonal modular invariants, the two pentagon relations admit an identical form and thus the quantum $3j$ and $6j$ symbols coincide up to a gauge [2, 6] confirming an earlier result in [7], where the three-point boundary functions were computed explicitly in the $sl(2)$ case.

These structures are considerably more complicated in the Liouville theory, a non-compact $c > 25$ Virasoro theory with representations (and possibly boundaries), described by a continuous spectrum. The study of the boundary Liouville theory started with the paper [8] by Fateev–Zamolodchikov–Zamolodchikov (FZZ), in which, in particular, the two-point function of boundary operators—a special case of the non-compact $6j$ symbols—was computed. The general boundary OPE coefficients with boundaries of the FZZ type were determined in [9] similarly to the diagonal rational theory by identifying them up to a gauge with the Liouville quantum $6j$ symbols computed in [10]. The pentagon equations in this case were further

discussed and used in [11], motivated by a comparison with the microscopic approach to the Liouville gravity.

In this work, we consider the non-critical string analogue of the boundary pentagon relations and their solutions⁴. The theory combines two Virasoro theories, $c < 1$ (matter) and $c > 25$ (Liouville), so that the overall central charge is compensated by the central charge of a pair of free ghost fields. As in the bulk [13, 14], the emphasis is on the presence of non-trivial matter interaction implemented conventionally by the two $c < 1$ screening charges. Our derivation here exploits only the factorization of the three-point tachyon boundary correlators into matter and Liouville factors. It yields equations which can be obtained alternatively in the ground ring approach [14–18], using the coefficients in the OPE of the ground ring generators and the tachyons. The result is a generalization of the trivial matter case considered in [11], in which the tachyon correlators are described by the correlators in the pure Liouville theory but with additional constraints on the set of representations arising from the mass-shell condition.

The solution of the general equations is a product of matter and Liouville three-point boundary coefficients. We consider the case when the matter fields are restricted by a charge conservation condition with two types of screening charges or/and correspond to degenerate $c < 1$ Virasoro representations. In this case, the matter factor is given by the Coulomb gas expression and in the non-rational case it can be recovered by analytic continuation of the $c > 25$ Liouville Coulomb gas expression. The Liouville factor of the tachyon three-point correlator is given in principle by the integral Ponsot–Teschner (PT) formula, which is however rather complicated; in contrast, simple meromorphic expressions were found for certain particular examples in the microscopic approach of [11]. This motivated us to derive a compact general expression for the Liouville boundary OPE coefficients using recursively the Liouville pentagon equations.

The new formula is valid for Liouville charges corresponding to degenerate matter representations and is expressed in terms of finite sum basic hypergeometric functions of type ${}_4\Phi_3$ with bounds generically consistent with the matter fusion rules. The formula generalizes a particular (thermal) case result of [19] obtained by formal manipulations of the PT integral; unlike the expression found in [19] in this particular case, our formula is explicitly invariant under cyclic permutations of the boundary fields.

2. Pentagon equations

We shall keep only the Liouville field labels for the tachyon boundary operator $T_\beta^{(\epsilon)} =_{(\bar{\sigma}_2, \sigma_2)} T_{(e, \beta)}^{(\bar{\sigma}_1, \sigma_1)}$ of chirality $\epsilon = \pm 1$, $(e, \beta) = (\epsilon\beta - \epsilon b^\epsilon, \beta)$, while the matter representation label e and matter boundary labels $\bar{\sigma}_i$ will be suppressed. The parameter b determining the central charges $c = 13 + 6(b^2 + 1/b^2) > 25$ and $c = 13 - 6(b^2 + 1/b^2) < 1$ of the two Virasoro theories is generically an arbitrary real number; most of the formulae below remain true for the rational (minimal matter) case. It is convenient to use the ‘leg factor’ normalization

$$T_\beta^{(\epsilon)}(x) = \Gamma(b^\epsilon(Q - 2\beta)) \mathbf{c}(x) e^{2i\epsilon(\beta - b^\epsilon)\chi(x)} e^{2\beta\phi(x)}. \quad (2.1)$$

The scaling dimensions are respectively given by

$$\begin{aligned} \Delta_L(\beta) &= \beta(Q - \beta), & Q &= 1/b + b, \\ \Delta_M(e) &= e(e - e_0), & e_0 &= 1/b - b, \\ \Delta_M(e) + \Delta_L(\epsilon e + b^\epsilon) &= 1 = -\Delta_{\text{ghost } \mathbf{c}}. \end{aligned} \quad (2.2)$$

⁴ This contribution (see also arXiv:0805.0134) is based on [12] (sections 2 and 3) and on the preprint of the authors (section 4), ‘Boundary three-point Liouville dressing factor for $c < 1$ degenerate matter’, preprint ESI 2060 (2008) available at <http://www.esi.ac.at/preprints/ESI-Preprints.html>.

- The pentagon relations take a simple recursive form when one of the operators corresponds to a fundamental degenerate Virasoro representation. Starting with the Liouville case, one has (see, e.g., [11])

$$\begin{aligned}
 & C_{\sigma_3, \beta_2 - t \frac{b}{2}}^L \begin{bmatrix} \beta_2 & -\frac{b}{2} \\ \sigma_4 & \sigma_2 \end{bmatrix} C_{\sigma_2 = \sigma_3 \pm \frac{b}{2}, \beta_3}^L \begin{bmatrix} \beta_2 - t \frac{b}{2} & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} \\
 &= F_{+,t}^L \begin{bmatrix} \beta_2 & -\frac{b}{2} \\ \beta_3 & \beta_1 \end{bmatrix} C_{\sigma_3 \pm \frac{b}{2}, \beta_1 - \frac{b}{2}}^L \begin{bmatrix} -\frac{b}{2} & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} C_{\sigma_3, \beta_3}^L \begin{bmatrix} \beta_2 & \beta_1 - \frac{b}{2} \\ \sigma_4 & \sigma_1 \end{bmatrix} \\
 &+ F_{-,t}^L \begin{bmatrix} \beta_2 & -\frac{b}{2} \\ \beta_3 & \beta_1 \end{bmatrix} C_{\sigma_3 \pm \frac{b}{2}, \beta_1 + \frac{b}{2}}^L \begin{bmatrix} -\frac{b}{2} & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} C_{\sigma_3, \beta_3}^L \begin{bmatrix} \beta_2 & \beta_1 + \frac{b}{2} \\ \sigma_4 & \sigma_1 \end{bmatrix}, \quad t = \pm 1. \quad (2.3)
 \end{aligned}$$

The relation of the OPE coefficients to the (cyclically symmetric) three-point correlators is

$$\begin{aligned}
 C_{\sigma_2, Q - \beta_3}^L \begin{bmatrix} \beta_2 & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} &= \langle \sigma_1 B_{\beta_3}^{\sigma_3} B_{\beta_2}^{\sigma_2} B_{\beta_1}^{\sigma_1} \rangle = {}^L C_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} \\
 &= S(\sigma_1, \beta_3, \sigma_3) {}^L C_{Q - \beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1}, \quad (2.4)
 \end{aligned}$$

where $S(\sigma_1, \beta_3, \sigma_3)$ is the reflection amplitude [8]. The Coulomb gas constants computed for labels $\{\beta_i\}$ restricted by the charge conservation condition $\sum_i \beta_i - Q = -mb - \frac{n}{b}$ (or any reflection of this condition) are recovered as residues from the expression in [9].

Similarly, the matter pentagon equation reads as

$$\begin{aligned}
 & C_{\bar{\sigma}_3, e_2 + t' \frac{b}{2}}^M \begin{bmatrix} e_2 & \frac{b}{2} \\ \bar{\sigma}_4 & \bar{\sigma}_2 \end{bmatrix} C_{\bar{\sigma}_2 = \bar{\sigma}_3 \mp \frac{b}{2}, e_3}^M \begin{bmatrix} e_2 + t' \frac{b}{2} & e_1 \\ \bar{\sigma}_4 & \bar{\sigma}_1 \end{bmatrix} \\
 &= F_{+, -t'}^M \begin{bmatrix} e_2 & \frac{b}{2} \\ e_3 & e_1 \end{bmatrix} C_{\bar{\sigma}_3 \mp \frac{b}{2}, e_1 - \frac{b}{2}}^M \begin{bmatrix} \frac{b}{2} & e_1 \\ \bar{\sigma}_3 & \bar{\sigma}_1 \end{bmatrix} C_{\bar{\sigma}_3, e_3}^M \begin{bmatrix} e_2 & e_1 - \frac{b}{2} \\ \bar{\sigma}_4 & \bar{\sigma}_1 \end{bmatrix} \\
 &+ F_{-, -t'}^M \begin{bmatrix} e_2 & \frac{b}{2} \\ e_3 & e_1 \end{bmatrix} C_{\bar{\sigma}_3 \mp \frac{b}{2}, e_1 + \frac{b}{2}}^M \begin{bmatrix} \frac{b}{2} & e_1 \\ \bar{\sigma}_3 & \bar{\sigma}_1 \end{bmatrix} C_{\bar{\sigma}_3, e_3}^M \begin{bmatrix} e_2 & e_1 + \frac{b}{2} \\ \bar{\sigma}_4 & \bar{\sigma}_1 \end{bmatrix}, \quad t' = \pm 1, \quad (2.5)
 \end{aligned}$$

and the matter constants C^M will be normalized to 1 for $e_1 + e_2 + (e_0 - e_3) - e_0 = 0$.

- The fusing matrix elements and the boundary OPE coefficients in (2.3) and (2.5) containing a fundamental Virasoro representation are known constants, which are recalled in appendix A. These matter and Liouville fusing matrix elements are related by analytic continuation. For example, for the choice of the chiralities of the three fields as (+, −, +)

$$\beta_3 = e_3 + b, \quad \beta_2 = -e_2 + 1/b, \quad \beta_1 = e_1 + b, \quad (2.6)$$

one has $F_{s,t}^M = F_{-s,t}^L$, $\tilde{F}_{s,t}^M = \tilde{F}_{-s,t}^L$, which implies the following identities:

$$\begin{aligned}
 & F_{+,+}^L F_{-,-}^M - F_{+,-}^L F_{-,+}^M = 0 = F_{-,+}^L F_{+,-}^M - F_{-,-}^L F_{+,+}^M, \\
 & -F_{+,+}^L F_{+,-}^M + F_{+,-}^L F_{+,+}^M = \frac{Q - 2\beta_1}{Q - 2\beta_2} = -F_{-,-}^L F_{-,+}^M + F_{-,+}^L F_{-,-}^M. \quad (2.7)
 \end{aligned}$$

Now we multiply the matter and Liouville pentagon identities (2.3) and (2.5) for the same fixed $t = t'$ which is consistent with a tachyon of negative chirality $(e_2 + t \frac{b}{2}, \beta_2 - t \frac{b}{2})$ in the lhs. On the other hand, in the rhs we also get two mixed terms besides the two tachyon contributions, inconsistent with the mass-shell condition. Due to the first of the identities in (2.7), these mixed terms are cancelled in the linear combination of the $t = +1$ and $t = -1$ product identities taken with a relative minus sign. To compute this linear combination, one

has to take into account the second identity (2.7) and one finally obtains for the normalized as in (2.1) tachyon OPE coefficients \hat{C} :

$$\begin{aligned} \hat{C}_{\sigma_2, \beta_3} \begin{bmatrix} \beta_2 - \frac{b}{2} & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} &+ \sqrt{\lambda_L \lambda_M} c(\beta_2) c_{(-\delta)}^M \left(\bar{\sigma}_2 = \bar{\sigma}_3 - \delta \frac{b}{2}, e_2, \bar{\sigma}_4 \right) \\ &\times c_{(\delta)}^L \left(\sigma_2 = \sigma_3 + \delta \frac{b}{2}, \beta_2, \sigma_4 \right) \hat{C}_{\sigma_2, \beta_3} \begin{bmatrix} \beta_2 + \frac{b}{2} & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} \\ &= -\sqrt{\lambda_M} c_{(\delta)}^M(\bar{\sigma}_3, e_1, \bar{\sigma}_1) \hat{C}_{\sigma_3, \beta_3} \begin{bmatrix} \beta_2 & \beta_1 - \frac{b}{2} \\ \sigma_4 & \sigma_1 \end{bmatrix} \\ &\quad - \sqrt{\lambda_L} c_{(-\delta)}^L(\sigma_3, \beta_1, \sigma_1) \hat{C}_{\sigma_3, \beta_3} \begin{bmatrix} \beta_2 & \beta_1 + \frac{b}{2} \\ \sigma_4 & \sigma_1 \end{bmatrix}, \end{aligned} \quad (2.8)$$

where $\delta, \bar{\delta} = \pm 1$,

$$c_{(\mp)}^L(\sigma_3, \beta_1, \sigma_1) = \frac{2 \sin \pi b (\beta_1 \mp (\sigma_1 + \sigma_3 - Q) - \frac{b}{2}) \sin \pi b (\beta_1 \mp (\sigma_3 - \sigma_1) - \frac{b}{2})}{\sin \pi b (Q - 2\beta_1)}, \quad (2.9)$$

$$c_{(\pm)}^M(\bar{\sigma}_3, e_1, \bar{\sigma}_1) = \frac{2 \sin \pi b (e_1 \mp (\bar{\sigma}_1 + \bar{\sigma}_3 - e_0) + \frac{b}{2}) \sin \pi b (e_1 \mp (\bar{\sigma}_3 - \bar{\sigma}_1) + \frac{b}{2})}{\sin \pi b (e_0 - 2e_1)} \quad (2.10)$$

and

$$c(\beta_2) = -\frac{\sin \pi b (Q - 2\beta_2)}{\sin \pi b (2\beta_2)}. \quad (2.11)$$

The constants λ_L, λ_M in (2.8) are the two bulk coupling constants, following the notation in [14]. Similarly, one obtains the dual equation with $\tilde{\lambda}_L = \lambda_L^{1/b^2}, \tilde{\lambda}_M = \lambda_M^{-1/b^2}$:

$$\begin{aligned} -\sqrt{\tilde{\lambda}_M} \tilde{c}_{(\delta)}^M(\bar{\sigma}_2, e_2, \bar{\sigma}_4) \hat{C}_{\sigma_2, \beta_3} \begin{bmatrix} \beta_2 - \frac{1}{2b} & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} &- \sqrt{\tilde{\lambda}_L} \tilde{c}_{(\delta)}^L(\sigma_2, \beta_2, \sigma_4) \hat{C}_{\sigma_2, \beta_3} \begin{bmatrix} \beta_2 + \frac{1}{2b} & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} \\ &= \hat{C}_{\sigma_3, \beta_3} \begin{bmatrix} \beta_2 & \beta_1 - \frac{1}{2b} \\ \sigma_4 & \sigma_1 \end{bmatrix} + \sqrt{\tilde{\lambda}_L \tilde{\lambda}_M} \tilde{c}(\beta_1) \tilde{c}_{(-\delta)}^M \left(\bar{\sigma}_3 = \bar{\sigma}_2 - \frac{\delta}{2b}, e_1, \bar{\sigma}_1 \right) \\ &\quad \times \tilde{c}_{(-\delta)}^L \left(\sigma_3 = \sigma_2 - \frac{\delta}{2b}, \beta_1, \sigma_1 \right) \hat{C}_{\sigma_3, \beta_3} \begin{bmatrix} \beta_2 & \beta_1 + \frac{1}{2b} \\ \sigma_4 & \sigma_1 \end{bmatrix}, \end{aligned} \quad (2.12)$$

replacing the constants in (2.9) (and (2.11)) with their duals, obtained by the change $b \rightarrow 1/b$ (for β_i -fixed), while the dual of the matter constant (2.10) is obtained with $b \rightarrow -1/b$, so that

$$\tilde{c}_{(\mp)}^M(\bar{\sigma}_3, e, \bar{\sigma}_1) = \frac{2 \sin \pi \frac{1}{b} (e - \frac{1}{2b} \mp (\bar{\sigma}_3 + \bar{\sigma}_1 - e_0)) \sin \pi \frac{1}{b} (e - \frac{1}{2b} \mp (\bar{\sigma}_3 - \bar{\sigma}_1))}{\sin \pi \frac{1}{b} (2e - e_0)}. \quad (2.13)$$

The two sets of equations (2.8) and (2.12) are precisely the equations one obtains starting from a four-point function with a ground ring generator being added and then inserting the coefficients in the expansion of the product of the ground ring generator with the left or right tachyons (see formulae (A.36)–(A.38) of [14]; the computation there completes earlier partial results [16–18] for these OPE coefficients).

2.1. Special case—trivial matter

We choose as before the chiralities of type (+ +). For trivial matter, i.e. a charge conservation condition with no screening charges

$$e_0 = e_1 + \left(e_2 + \frac{b}{2} \right) + (e_0 - e_3) \equiv e_{12}^3 + e_0 + \frac{b}{2} \Rightarrow \beta_{23}^1 + \frac{b}{2} = Q, \quad (2.14)$$

the matter boundary three-point functions are trivial and the pure Liouville identity (2.3) ($t = +1$) (normalized with the leg factors) with β_i restricted by (2.14) simplifies to

$$\begin{aligned} \hat{C}_{\sigma_3 \pm \frac{b}{2}, \beta_3} \begin{bmatrix} \beta_2 - \frac{b}{2} & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} &= -\sqrt{\lambda_L} c_{(\mp)}^L(\sigma_3, \beta_1, \sigma_1) \hat{C}_{\sigma_3, \beta_3} \begin{bmatrix} \beta_2 & \beta_1 + \frac{b}{2} \\ \sigma_4 & \sigma_1 \end{bmatrix} \\ &+ F_{+,+}^L \Gamma\left(\frac{1}{b}(Q - 2\beta_2 + b)\right) \Gamma(b(Q - 2\beta_1)) \Gamma(b(2\beta_3 - Q)) C_{\sigma_3, \beta_3}^L \begin{bmatrix} \beta_2 & \beta_1 - \frac{b}{2} \\ \sigma_4 & \sigma_1 \end{bmatrix} \\ &= -\sqrt{\lambda_L} c_{(\mp)}^L(\sigma_3, \beta_1, \sigma_1) \hat{C}_{\sigma_3, \beta_3}^L \begin{bmatrix} \beta_2 & \beta_1 + \frac{b}{2} \\ \sigma_4 & \sigma_1 \end{bmatrix} + \frac{2\pi G_2(\sigma_3, \beta_2, \sigma_4)}{2 \sin(\pi b(Q - 2\beta_1))}. \end{aligned} \quad (2.15)$$

We have used the fact that for the values in (2.14) F_{++}^L has a zero $\sum_i \beta_i' - Q \rightarrow 0$, while the Liouville-reflected three-point constant has a singularity with residue $1/2\pi$. Thus, the second term in (2.15) reduces to the (leg-normalized) reflection Liouville amplitude [8]:

$$\begin{aligned} \Gamma\left(\frac{1}{b}(Q - 2\beta)\right) \Gamma(b(Q - 2\beta)) S(\sigma_2, \beta, \sigma_1) &= \frac{2\pi}{Q - 2\beta} G_2(\sigma_2, \beta, \sigma_1), \\ G_2(\sigma_2, \beta, \sigma_1) &= \frac{\lambda_L^{\frac{1}{2b}(Q-2\beta)} S_b(2\beta - Q)}{\prod_{s=\pm} S_b(\beta + s(\sigma_2 + \sigma_1 - Q)) S_b(\beta + s(\sigma_2 - \sigma_1))}, \\ G_2(\sigma_2, \beta, \sigma_1) G_2(\sigma_2, Q - \beta, \sigma_1) &= S_b(2\beta - Q) S_b(Q - 2\beta), \end{aligned} \quad (2.16)$$

where $S_b(x) = \Gamma_b(x) / \Gamma_b(Q - x)$ and $\Gamma_b(x)$ is the double Gamma function. The amplitude G_2 , which can be identified with the tachyon two-point function [19], is the solution of the equations

$$\begin{aligned} -\sqrt{\lambda_L} c_{(\mp)}^L(\sigma_3, \beta_1, \sigma_1) G_2\left(\sigma_3, \beta_1 + \frac{b}{2}, \sigma_1\right) &= G_2\left(\sigma_3 \pm \frac{b}{2}, \beta_1, \sigma_1\right), \\ -\sqrt{\tilde{\lambda}_L} \tilde{c}_{(\mp)}^L(\sigma_3, \beta_1, \sigma_1) G_2\left(\sigma_3, \beta_1 + \frac{1}{2b}, \sigma_1\right) &= G_2\left(\sigma_3 \pm \frac{1}{2b}, \beta_1, \sigma_1\right). \end{aligned} \quad (2.17)$$

Since the general identity (2.8) should reduce to the simpler identity (2.15) for the values in (2.14), this implies a restriction on the unknown matter OPE coefficients involved in (2.8).

The identity (2.15) acquires a more symmetric form when rewritten for the cyclically symmetric correlator of type $(- - -)^5$ obtained by two reflections (2.4), now written for the normalized correlators

$$\begin{aligned} \hat{C}_{Q-\beta_3, \beta_2, \beta_1}^{\sigma_4, \sigma_2, \sigma_1} &= \frac{1}{b} \frac{1}{2 \sin(\pi b(2\beta_3 - Q))} G_2^{-1}(\sigma_1, \beta_3, \sigma_4) \hat{C}_{\beta_3, \beta_2, \beta_1}^{\sigma_4, \sigma_2, \sigma_1} \\ &= \frac{1}{b^2} \frac{\sin\left(\pi \frac{1}{b}(Q - 2\beta_1)\right)}{\sin(\pi b(Q - 2\beta_3))} \frac{G_2(\sigma_2, \beta_1, \sigma_1)}{G_2(\sigma_1, \beta_3, \sigma_4)} \hat{C}_{\beta_3, \beta_2, Q-\beta_1}^{\sigma_4, \sigma_2, \sigma_1}. \end{aligned} \quad (2.18)$$

For comparison, we give equations analogous to (2.17) for the nontrivial matter two-point amplitude:

$$\begin{aligned} -\sqrt{\lambda_M} c_{(\mp)}^M(\bar{\sigma}_3, e, \bar{\sigma}_1) G_2^M\left(\bar{\sigma}_3, e - \frac{b}{2}, \bar{\sigma}_1\right) &= G_2^M\left(\bar{\sigma}_3 \pm \frac{b}{2}, e, \bar{\sigma}_1\right), \\ -\sqrt{\tilde{\lambda}_M} \tilde{c}_{(\mp)}^M(\bar{\sigma}_3, e, \bar{\sigma}_1) G_2^M\left(\bar{\sigma}_3, e + \frac{1}{2b}, \bar{\sigma}_1\right) &= G_2^M\left(\bar{\sigma}_3 \pm \frac{1}{2b}, e, \bar{\sigma}_1\right). \end{aligned} \quad (2.19)$$

⁵ This equation has been independently written down recently in [20].

3. The matter factor in the solution

We shall start with the solution of the two-point equations (2.19) for the matter degenerate values $2e = mb - n/b =: 2e_{m,n}$, where m, n are nonnegative integers, $m, n \in \mathbb{Z}_{\geq 0}$. The solution is conveniently expressed as

$$\begin{aligned}
 G_2^M(\bar{\sigma}_2, e, \bar{\sigma}_1) &= \frac{(-1)^{(m+1)(n+1)} \lambda_M^{\frac{2e-\epsilon_0}{2b}}}{S_b((m+2)b)S_b(\frac{n+2}{b})} \frac{G_M(\bar{\sigma}_1, e - b - (m+1)b, \bar{\sigma}_2)}{G_M(\bar{\sigma}_1, e - b + \frac{n+1}{b}, \bar{\sigma}_2)} \\
 &= \lambda_M^{\frac{2e-\epsilon_0}{2b}} \lambda_L^{-\frac{(Q+mb+n/b)}{2b}} G_2\left(\bar{\sigma}_2 + b, -e - \frac{n}{b}, \bar{\sigma}_1 + b\right) \frac{S_b(2Q + mb + \frac{n}{b})S_b(\frac{1}{b})}{S_b^2(\frac{n+2}{b})},
 \end{aligned} \tag{3.1}$$

where

$$G_M(\bar{\sigma}_3, e_2, \bar{\sigma}_2) := S_b(-e_2 + \bar{\sigma}_2 + \bar{\sigma}_3)S_b(e_0 - e_2 + \bar{\sigma}_3 - \bar{\sigma}_2). \tag{3.2}$$

The representation of (3.1) in terms of $S_b(x)$ is not unique, but the expression is finite for the concrete values $e = e_{m,n}$ and reduces to a finite product of sines. Equations (2.19) allow us to extend formula (3.1) to $m = n = -1$ and furthermore to the degenerate values $2e = e_0 - (m+1)b + \frac{n+1}{b}$ with $m, n \in \mathbb{Z}_{\geq 0}$.⁶

- The solution of the pair of equations (2.8) and (2.12) is given by a factorized expression combining the known Liouville expression [9] and a solution of the matter boundary pentagon equation. The solution of the matter boundary pentagon equation is a generalization to generic b^2 of the solution in the rational $c < 1$ case, where the fusing matrix is given [21] by a product of two basic ${}_4\Phi_3$ hypergeometric functions known to represent [22] the quantum 6j symbols. The change of gauge affects only the prefactor. The non-rational generalization is possible either if the representations are chosen to correspond to degenerate $c < 1$ Virasoro representations or if a charge conservation condition with integer numbers of matter screening charges is imposed; we refer to both as ‘Coulomb gas’ cases. The solutions in these cases are alternatively reproduced starting from the general formula of [9]. Thus to obtain the matter constant for

$$e_{123} - e_0 \equiv e_1 + e_2 + e_3 - e_0 = mb - n/b, \quad m, n \in \mathbb{Z}_{\geq 0}, \tag{3.3}$$

we start from the Liouville Coulomb gas expression for $\alpha_{123} - Q = -mb - n/b$ derived as a residue of the formula in [9]. We rewrite this particular solution of the Liouville pentagon equation (2.3) in terms of finite products of Gamma and sine functions and then continue analytically the result by replacing $b^2 \rightarrow -b^2$ and $\alpha_i b \rightarrow e_i b$. The final result is a solution of the matter pentagon equation (2.5) and can again be expressed in a compact form in terms of the ratios of double Gamma functions $\Gamma_b(x)$ using the notation (3.2):

$$\begin{aligned}
 C_{\bar{\sigma}_2, e_0 - e_3}^M \begin{bmatrix} e_2 & e_1 \\ \bar{\sigma}_3 & \bar{\sigma}_1 \end{bmatrix} &= {}^M C_{e_3, e_2, e_1}^{\bar{\sigma}_3, \bar{\sigma}_2, \bar{\sigma}_1} = (-1)^{m+n} \lambda_M^{\frac{e_{123} - e_0}{2b}} \Pi_M(e_3, e_2, e_1) \\
 &\times \frac{(-1)^{mn} S_b(b + 2e_1 - mb)}{S_b(b + 2e_1 + \frac{n}{b})} \\
 &\times \sum_{k=0}^m \sum_{p=0}^n \frac{G_M(\bar{\sigma}_3, e_2 - b - kb, \bar{\sigma}_2) G_M(\bar{\sigma}_3, e_0 - e_3 - b - \frac{n-p}{b}, \bar{\sigma}_1)}{G_M(\bar{\sigma}_3, e_0 - e_3 - b + (m-k)b, \bar{\sigma}_1) G_M(\bar{\sigma}_3, e_2 - b + \frac{p}{b}, \bar{\sigma}_2)} \\
 &\times \frac{S_b(b + 2e_3 - (m-k)b) S_b(b + 2e_2 - kb) S_b(\frac{1}{b} - 2e_2 - \frac{p}{b}) S_b(\frac{1}{b} - 2e_3 - \frac{n-p}{b})}{S_b((k+1)b) S_b((m-k+1)b) S_b(\frac{p+1}{b}) S_b(\frac{n-p+1}{b})},
 \end{aligned} \tag{3.4}$$

⁶ Note that unlike the Liouville case the analytic continuation of the two thermal cases $n = 0$ or $m = 0$ to generic values of e leads to different results, effectively inverse to each other.

where

$$\begin{aligned} \Pi_M(e_3, e_2, e_1) &= \frac{b^{Q(e_{123}-e_0)} \Gamma_b(b) S_b\left(\frac{n+1}{b}\right)}{\Gamma_b\left(\frac{1}{b} + e_0 - e_{123}\right)} \prod_i \frac{\Gamma_b\left(\frac{1}{b} - 2e_i\right) S_b\left(b + 2e_i + \frac{n}{b}\right)}{\Gamma_b(b + 2e_i + e_0 - e_{123})} \\ &= \frac{b^{Q(e_{123}-e_0)} \Gamma_b\left(\frac{1}{b}\right) S_b(m+1)b}{\Gamma_b(b - e_0 + e_{123})} \prod_i \frac{\Gamma_b(b + 2e_i) S_b\left(\frac{1}{b} - 2e_i + mb\right)}{\Gamma_b\left(\frac{1}{b} - 2e_i - e_0 + e_{123}\right)}. \end{aligned} \quad (3.5)$$

This formula is derived for generic values of $\{e_i\}$, subject to the constraint (3.3), but it reproduces as well the constants with degenerate values of e_i .

4. The Liouville three-point dressing correlator

The matter charge conservation condition (3.3) can be rewritten as a relation for the Liouville labels, e.g. with the choice of chiralities (+ - +), one has

$$\beta_{13}^2 \equiv \beta_1 + \beta_3 - \beta_2 = (m+1)b - \frac{n}{b}, \quad m, n \in \mathbb{Z}_{\geq 0}. \quad (4.1)$$

In addition, we also choose degenerate values for all matter labels or, equivalently, in terms of the Liouville labels β_i of the three fields in the correlator we take

$$\beta_i = b + m_i b - \frac{n_i}{b}, \quad 2m_i, 2n_i \in \mathbb{Z}_{\geq 0}. \quad (4.2)$$

We further impose the (matter) fusion rule restriction such that all $m_{ij}^k, n_{ij}^k, i \neq j \neq k \neq i$ are non-negative integers, so that $\sum_{i=1}^3 2m_i = 0 \pmod 2$. Other possible choices correspond to Liouville reflections $Q - \beta_i$ of some of the labels in (4.2), and the corresponding three-point correlator is obtained with the help of the reflection relation (2.18).

For such values of $\{\beta_i\}$, the integral PT formula for the Liouville three-point boundary constant simplifies. Taking into account two infinite series of poles it is rewritten as a sum of two terms, each expressed in terms of a product of basic ${}_4\Phi_3$ hypergeometric functions, one given by a finite (of range n as in (4.1)) and the other by an infinite sum. A resummation of the infinite sums was performed in [19] in the particular case $m_i = 0, i = 1, 2, 3$, of (4.2).⁷

Here, we shall follow a different route to obtain a general simple formula without exploiting the integral PT representation, namely we shall use recursively the Liouville pentagon equations (2.3).

- We start with the derivation of the simplest correlator with three identical charges equal to b , i.e. the correlator of three cosmological operators, or boundary Liouville screening charges. It is reproduced by the second term in the rhs of equality (2.3) choosing $\beta_1 = \frac{b}{2} = \beta_2, \beta_3 = b$. For this choice, the equation needs regularization since the coefficient in front of the correlator becomes divergent. The remaining two correlators are represented as reflections (2.4) with respect to β_3 (the lhs) and β_2 (the first term in the rhs) of correlators, which also diverge if we assume that they are given by the integral PT formula. Indeed they satisfy the charge conservation conditions $(Q - \beta_3) + \beta_2 + \beta_1 = Q$ and $\beta_3 + (Q - \beta_2 - b/2) + (\beta_1 - b/2) = Q$, respectively, and their residua equal $1/2\pi$ (to agree with the normalization in [9]); the reasoning here is similar to that in the derivation of the special case equation (2.15). Thus, in a proper regularization of (2.3), these two correlators are replaced by the corresponding reflection amplitudes, which appear as the

⁷ The formal resummation in [19], which we believe is correct only when applied to the sum of the two terms, amounts to a relation for ${}_3\Phi_2 q$ Saalschutz-type functions.

initial data in the equation. In the case under consideration $\beta = b$ and inserting in (2.3), we reproduce the cyclically symmetric expression of [11, 19]:

$$\begin{aligned} \hat{C}_{b,b,b}^{\sigma_3,\sigma_2,\sigma_1} &= \frac{2\pi\sqrt{\lambda_L}^{-1}}{g_-(\sigma_1, b/2, \sigma_2)} (G_2(\sigma_3, b, \sigma_1) - G_2(\sigma_3, b, \sigma_2)) \\ &= \frac{2\pi\lambda_L^{\frac{Q-3b}{2b}}}{S_b(\frac{2}{b})} \frac{(\tilde{c}_1(c_2 - c_3) + \tilde{c}_2(c_3 - c_1) + \tilde{c}_3(c_1 - c_2))}{(c_2 - c_1)(c_1 - c_3)(c_3 - c_2)}, \end{aligned} \quad (4.3)$$

where the boundary cosmological constants $\lambda_B = -\sqrt{\lambda_L}c(\sigma)$ and their dual appear:

$$\begin{aligned} c_i &= 2 \cos \pi b(b - 2\sigma_i), & \tilde{c}_i &= 2 \cos \pi \frac{1}{b} \left(\frac{1}{b} - 2\sigma_i \right), \\ g_-(\sigma_1, \beta, \sigma_2) &= 2 \cos \pi b(2\beta - 2\sigma_1) - 2 \cos \pi b(b - 2\sigma_2). \end{aligned} \quad (4.4)$$

4.1. One-parameter correlators, cyclic symmetry

Let us first consider the ‘thermal’ case with all $n_i = 0$ in (4.2). The Liouville correlator in (4.3) (normalized with the leg factors (2.1)) coincides with the tachyon correlator itself since it corresponds to a trivial matter condition with $m = 0 = n$ in (3.3) and (4.1). Applying first trivial matter equations of the type in (2.15), we get the most general correlator with $m_{13}^2 = 0$. Then using the general equation (2.3) (for shifts of the pair (β_3, β_2)), we obtain, denoting $m = m_{13}^2, s = m_{12}^3$,

$$\begin{aligned} {}^L C_{\beta_3,\beta_2,\beta_1}^{\sigma_3,\sigma_2,\sigma_1} &= \lambda_L^{-1/2} \Pi_L(\beta_3, \beta_2, \beta_1) S_b \left(\frac{2}{b} \right) \left(\frac{G_2(\sigma_2 + b, \beta_1, \sigma_1)}{g_-(\sigma_2, \beta_1 - \frac{b}{2}, \sigma_1)} \frac{S_b((s+1)b)}{S_b((s+m+1)b)} \right. \\ &\quad \times \sum_{p=0}^s \frac{S_b((p+m)b+Q)}{S_b(pb+Q)} \frac{G_2(\sigma_2 + p\frac{b}{2}, \beta_2 - p\frac{b}{2}, \sigma_3)}{G_2(\sigma_2 + b + (p+m)\frac{b}{2}, \beta_1 - (p+m)\frac{b}{2}, \sigma_1)} \\ &\quad + \frac{G_2(\sigma_1 - b, \beta_1, \sigma_2)}{g_-(\sigma_1, Q - \beta_1 + \frac{b}{2}, \sigma_2)} \frac{S_b((m+1)b)}{S_b((s+m+1)b)} \\ &\quad \left. \times \sum_{r=0}^m \frac{S_b((r+s)b+Q)G_2(\sigma_1 - r\frac{b}{2}, \beta_3 - r\frac{b}{2}, \sigma_3)}{S_b(rb+Q)G_2(\sigma_1 - b - (r+s)\frac{b}{2}, \beta_1 - (r+s)\frac{b}{2}, \sigma_2)} \right), \end{aligned} \quad (4.5)$$

where

$$\Pi_L(\beta_3, \beta_2, \beta_1) = \frac{b^{e_0(Q-\beta_{123})}\Gamma_b(2Q-\beta_{123})\Gamma_b(Q-\beta_{23}^1)\Gamma_b(Q-\beta_{13}^2)\Gamma_b(Q-\beta_{12}^3)}{S_b(\frac{1}{b})S_b(\frac{2}{b})\Gamma_b(Q)\Gamma_b(Q-2\beta_1)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_3)}. \quad (4.6)$$

In the overall prefactor in the product of the Liouville and matter correlators combining (3.5) and (4.6) and the leg factor normalization (2.1), the Γ_b functions are fully compensated; for example, with the choice of the chiralities (+, -, +), one has

$$\begin{aligned} &\Gamma(b(Q-2\beta_3))\Gamma\left(\frac{1}{b}(Q-2\beta_2)\right)\Gamma(b(Q-2\beta_1))\Pi_M\left(\beta_3-b, -\beta_2+\frac{1}{b}, \beta_1-b\right)\Pi_L(\beta_3, \beta_2, \beta_1) \\ &= \frac{2\pi S_b(2\beta_1-b)S_b(2\beta_2-\frac{1}{b})S_b(2\beta_3-b)}{S_b(2\beta_1-b-mb)S_b(2\beta_2-\frac{1}{b}-\frac{n}{b})S_b(2\beta_3-b-mb)} \frac{S_b(\frac{n+1}{b})}{S_b(\frac{1}{b})S_b(\frac{2}{b})}. \end{aligned} \quad (4.7)$$

In appendix C, we give a few explicit examples demonstrating two formulae (3.4) and (4.5). We shall now rewrite (4.5) in a form which reveals its symmetry under cyclic permutations. Let us first introduce some general notation

$$G^{(-)}(\sigma_2, \beta, \sigma_1) := S_b(-\beta + \sigma_2 + \sigma_1)S_b(Q - \beta + \sigma_2 - \sigma_1) = (G^{(+)}(\sigma_2, Q - \beta, \sigma_1))^{-1},$$

$$\frac{G^{(\pm)}(\sigma_2, \beta - \frac{b}{2}, \sigma_1)}{G^{(\pm)}(\sigma_2, \beta + \frac{b}{2}, \sigma_1)} = g_{\pm}(\sigma_2, \beta, \sigma_1) = 2 \sin \pi b(Q - 2\beta)c_{\pm}^L(\sigma_2, \beta, \sigma_1). \quad (4.8)$$

For a non-negative integer k and an integer n of parity $p(n)$, denote

$$B(\sigma_2, \sigma_1)^{(k;p(n))} := \frac{G^{(-)}(\sigma_2, -\frac{kb}{2} - \frac{n}{2b}, \sigma_1)}{G^{(-)}(\sigma_2, b + \frac{kb}{2} - \frac{n}{2b}, \sigma_1)} = (-1)^{(k+1)(n+1)} B(\sigma_1, \sigma_2)^{(k;p(n))}, \quad (4.9)$$

which is expressed as a $k + 1$ order polynomial in $\{c_i\}$ using that for $k \neq 0$

$$g_- \left(\sigma_2, \frac{b}{2} - k\frac{b}{2} + \frac{n}{2b}, \sigma_1 \right) g_- \left(\sigma_2, \frac{b}{2} + k\frac{b}{2} + \frac{n}{2b}, \sigma_1 \right) = c_1^2 + c_2^2 - c_1 c_2 (-1)^n 2 \cos \pi k b^2 - (2 \sin \pi k b^2)^2$$

while $B(\sigma_2, \sigma_1)^{(0;p(n))} = (-1)^n c_2 - c_1$. Similarly, one defines the dual $\tilde{B}(\sigma_2, \sigma_3)^{(n;p(k))}$ so that the reflection amplitude is expressed as the ratio

$$\frac{\lambda_L^{\frac{2\beta_2-Q}{2b}} G_2(\sigma_2, \beta_2 = b + m_2 b - n_2/b, \sigma_3)}{S_b(2\beta_2 - Q)} = \frac{G^{(-)}(\sigma_2, \beta_2, \sigma_3)}{G^{(-)}(\sigma_2, Q - \beta_2, \sigma_3)} = \frac{\tilde{B}(\sigma_2, \sigma_3)^{(2n_2;p(2m_2))}}{B(\sigma_2, \sigma_3)^{(2m_2;p(2n_2))}}. \quad (4.10)$$

Finally, we introduce

$$P_2 \equiv P_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} := (-1)^{m_{12}^3 + 2m_2} \lambda_L^{-\frac{m_{12}^3}{2}} \frac{S_b((2m_1 + 1)b)S_b((2m_2 + 1)b)}{S_b(b)}$$

$$\times \sum_{p=0}^{m_{12}^3} \frac{S_b((m_{12}^3 + 1)b)}{S_b((p + 1)b)S_b((m_{12}^3 + 1 - p)b)} \frac{G_2(\sigma_2 + p\frac{b}{2}, \beta_2 - p\frac{b}{2}, \sigma_3)}{G_2(\sigma_2, \beta_2, \sigma_3)}$$

$$\times \frac{G_2(\sigma_2 - (m_{12}^3 - p)\frac{b}{2}, \beta_1 - (m_{12}^3 - p)\frac{b}{2}, \sigma_1)}{G_2(\sigma_2, \beta_1, \sigma_1)}$$

$$= \frac{(-1)^{2m_2} S_b((m_{12}^3 + 1)b)}{S_b(b)} \sum_{p=0}^{m_{12}^3} \frac{S_b((2m_1 + p - m_{12}^3 + 1)b)S_b((2m_2 - p + 1)b)}{S_b((p + 1)b)S_b((m_{12}^3 + 1 - p)b)}$$

$$\times \frac{G^{(-)}(\sigma_2, \beta_2 - pb, \sigma_3)}{G^{(-)}(\sigma_2, \beta_2, \sigma_3)} \frac{G^{(+)}(\sigma_2, \beta_1 - (m_{12}^3 - p)b, \sigma_1)}{G^{(+)}(\sigma_2, \beta_1, \sigma_1)} \quad (4.11)$$

and similarly P_1 and P_3 , which can be obtained from (4.11) by cyclic permutations. The finite sum (4.11) is proportional to a truncated ${}_4\Phi_3$ -type function. It can be expanded as a polynomial in the variables $\{c_i\}$.

With this notation, (4.5) is cast in a form generalizing the second line in (4.3):

$$L C_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} = - \frac{\lambda_L^{\frac{Q-\beta_{123}}{2b}} \Pi_L(\beta_3, \beta_2, \beta_1)}{B(\sigma_1, \sigma_2)^{(2m_1;0)} B(\sigma_2, \sigma_3)^{(2m_2;0)} B(\sigma_3, \sigma_1)^{(2m_3;0)}} F_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1},$$

$$F_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} = (-1)^{2m_1} ((-1)^{2m_2} \tilde{c}_2 - \tilde{c}_3) B(\sigma_3, \sigma_1)^{(2m_3;0)} P_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1}$$

$$- (-1)^{2m_2} ((-1)^{2m_3} \tilde{c}_3 - \tilde{c}_1) B(\sigma_2, \sigma_3)^{(2m_2;0)} P_{\beta_2, \beta_1, \beta_3}^{\sigma_2, \sigma_1, \sigma_3}$$

$$= \tilde{c}_1 B(\sigma_3, \sigma_2)^{(2m_2;0)} P_{\beta_2, \beta_1, \beta_3}^{\sigma_2, \sigma_1, \sigma_3} + \tilde{c}_2 B(\sigma_1, \sigma_3)^{(2m_3;0)} P_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1}$$

$$+ \tilde{c}_3 B(\sigma_2, \sigma_1)^{(2m_1;0)} P_{\beta_1, \beta_3, \beta_2}^{\sigma_1, \sigma_3, \sigma_2}. \quad (4.12)$$

In the second equality of (4.12), we have exploited the relation

$$B(\sigma_3, \sigma_1)^{(2m_3;0)} P_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} + \text{cyclic permutations} = 0, \quad (4.13)$$

which is equivalent to the cyclic symmetry of the correlator, now explicit in (4.12). The symmetry is ensured by the fact that the expression given by the first equality satisfies all the equations related by cyclic permutations.

The composition of the reflection of all three fields with the reflection amplitude as in (2.4) and the duality transformation $b \rightarrow 1/b$ (changing notation $m_i \rightarrow n_i$) gives a correlator in the other thermal case when all $m_i = 0$ in (4.2). In that case, the product of $B^{(0; p(2n_i))}$ replaces the denominator in (4.12) and the formula confirms the structure suggested in the microscopic approach of [11]. The dual polynomial $\tilde{P}_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1}$ is defined by changing in (4.11) $\beta_i \rightarrow Q - \beta_i, b \rightarrow 1/b, m_i \rightarrow n_i$. With the help of some identities for the basic hypergeometric functions, one reproduces the formula found in [19] for the case $\{m_i = 0, n_i\text{-integers}\}$ by exploiting the PT formula in a formal way. The expression in [19] is not explicitly symmetric under cyclic permutations; rather, this symmetry is checked to hold on examples.

4.2. The general correlator

To obtain the Liouville correlator defined for the general values (4.2), one can either use the dual pentagon equations or start from the correlator with all $m_i = 0$. In one of the steps, the special case equation (2.15) has to be extended so that the second term in the rhs is given by G_2 times a non-trivial Coulomb gas Liouville correlator. The final result is an expression generalizing the first line in (4.12):

$$\begin{aligned} L C_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} = & - \frac{\lambda_L^{\frac{Q-\beta_{123}}{2b}} \Pi'_L(\beta_3, \beta_2, \beta_1)}{B(\sigma_1, \sigma_2)^{(2m_1; p(2n_1))} B(\sigma_2, \sigma_3)^{(2m_2; p(2n_2))} B(\sigma_3, \sigma_1)^{(2m_3; p(2n_3))}} (-1)^{2m_2 2n_1} \\ & \times \left((-1)^{2m_1+2n_2} \tilde{B}(\sigma_2, \sigma_3)^{(2n_2; p(2m_2))} \tilde{P}_{\beta_2, \beta_1, \beta_3}^{\sigma_2, \sigma_1, \sigma_3} B(\sigma_3, \sigma_1)^{(2m_3; p(2n_3))} P_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} \right. \\ & \left. - (-1)^{2m_2+2n_1} \tilde{B}(\sigma_3, \sigma_1)^{(2n_3; p(2m_3))} \tilde{P}_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} B(\sigma_2, \sigma_3)^{(2m_2; p(2n_2))} P_{\beta_2, \beta_1, \beta_3}^{\sigma_2, \sigma_1, \sigma_3} \right), \quad (4.14) \end{aligned}$$

$$\Pi'_L(\beta_3, \beta_2, \beta_1) = \frac{(-1)^{m_{123} n_{123}} \Pi_L(\beta_3, \beta_2, \beta_1) S_b^3\left(\frac{1}{b}\right) S_b\left(\frac{2}{b} - b\right)}{S_b\left(\frac{n_{12}^3+1}{b}\right) S_b\left(\frac{n_{23}^3+1}{b}\right) S_b\left(\frac{n_{31}^3+1}{b}\right) S_b\left(\frac{n_{123}^3+2}{b} - b\right)}. \quad (4.15)$$

Here, say, the polynomial P_2 is given by the first formula (4.11), where now all β_i are given by (4.2), with only the sign in front of (4.11) modified to $(-1)^{m_{12}^3(1+2n_3)+2m_3 2n_3+2m_2} = (-1)^{m_{123}(1+2n_3)+2m_1}$. Let us also write down the expression for one of the dual polynomials:

$$\begin{aligned} \tilde{P}_1 \equiv \tilde{P}_{\beta_2, \beta_1, \beta_3}^{\sigma_2, \sigma_1, \sigma_3} = & \frac{(-1)^{n_{123}(1+2m_2)+2n_3} \tilde{\lambda}_L^{-n_{13}^2/2} S_b\left(\frac{2n_1+1}{b}\right) S_b\left(\frac{2n_3+1}{b}\right)}{S_b\left(\frac{1}{b}\right)} \sum_{u=0}^{n_{13}^2} \frac{S_b\left(\frac{n_{13}^2+1}{b}\right)}{S_b\left(\frac{1+u}{b}\right) S_b\left(\frac{n_{13}^2+1-u}{b}\right)} \\ & \times \frac{G_2\left(\sigma_1 + \frac{u}{2b}, Q - \beta_1 - \frac{u}{2b}, \sigma_2\right) G_2\left(\sigma_1 - \frac{n_{13}^2-u}{2b}, Q - \beta_3 - \frac{n_{13}^2-u}{2b}, \sigma_3\right)}{G_2(\sigma_1, Q - \beta_1, \sigma_2) G_2(\sigma_1, Q - \beta_3, \sigma_3)}. \quad (4.16) \end{aligned}$$

Formula (4.14) gives the general expression for the Liouville factor in the tachyon three-point boundary correlator with degenerate $c < 1$ representations. The cyclic symmetry of the full correlator is ensured by construction and is equivalent to a relation generalizing (4.13)

$$(-1)^{2n_2(2m_2+1)} B(\sigma_3, \sigma_1)^{(2m_3; p(2n_3))} P_2 + \text{cyclic permutations} = 0 \quad (4.17)$$

and its dual with the dual polynomials and $m_i \leftrightarrow n_i$. In particular when all $m_i = 0$ the dual relation reproduces the cyclic identity satisfied by the first-order dual polynomials

$\tilde{B}(\sigma_2, \sigma_3)^{(0;p(2m_2))} = (-1)^{2m_2} \tilde{c}_2 - \tilde{c}_3$, etc, which appear in the numerator in (4.12). The composition of duality transformation $b \rightarrow 1/b, m_i \leftrightarrow n_i$ with reflection of all three fields keeps (4.14) invariant.

This solution of the Liouville pentagon equations extends to the minimal gravity theory with rational b^2 , in which case there may appear further truncations of the sums.

5. Summary and concluding remarks

We have derived the basic pentagon like equations (2.8) and (2.12) for the tachyon boundary OPE coefficients from the analogous equations for the matter and Liouville constituents, respectively. Their solutions were described for degenerate matter $c < 1$ representations with generic real values of the parameter b .

One of the main results of this work is the new expression for the general Liouville dressing factor for this range of representations, i.e. for values (4.2) of the Liouville charges β_i and their reflection images. Formula (4.14) represents the Liouville correlator as a ratio of polynomials of the boundary cosmological parameters c_i, \tilde{c}_i generalizing partial results in [11, 19]. The polynomials P_i in the numerator of the Liouville factor are given by basic hypergeometric functions which coincide, with proper identification of the parameters, with those appearing in the matter three-point function (3.4). More precisely, the (thermal) matter three-point function (3.4) is represented as

$${}^M C_{e_3, e_2, e_1}^{\tilde{\sigma}_3, \tilde{\sigma}_2, \tilde{\sigma}_1} \sim P_{\frac{1}{b} - \tilde{\sigma}_1, \frac{1}{b} - \tilde{\sigma}_3, \frac{1}{b} - \tilde{\sigma}_2}^{\frac{1}{b} - e_1, e_3 + b, e_2 + b} = P_3. \tag{5.1}$$

Similarly for $e_1 = m_1 b, e_3 = m_3 b, e_2 = e_0 - m_2 b$ (3.4) is identified with the polynomial P_1 in (4.12), etc. Analogous to (5.1) formulae hold for the case $\beta_i = b - n_i/b$, relating (3.4) to one of the dual polynomials with $\sigma_i = \tilde{\sigma}_i + b$.

The tachyon boundary correlator, being a product of matter and Liouville correlators, depends on ‘too many’ boundaries—their cardinality should be the same as that of the set of tachyons. Examples of a linearly independent set of boundaries are provided by the ‘trivial matter boundaries’, i.e. one σ_i is set to zero, while the intermediate two are fixed by the fusion rules; the matter factor is reduced to a correlator of chiral vertex operators. On the level of bulk one-point functions or boundary states, the states $|\sigma; \bar{\sigma}\rangle$ with general degenerate matter boundaries $\bar{\sigma}$ are represented as linear combinations of states with trivial matter $\bar{\sigma} = 0$ but shifted Liouville boundary parameters. This fusion-like relation can be lifted to the boundary three-point tachyon correlators by using recursively the identities (2.8) and (2.12); see also the recent work [23] for an explicit operator relation of this type.

Generically, the values of the boundary parameters correspond to the FZZ values with pure imaginary $Q - 2\sigma_i$. Another choice, the consistency of which deserves to be investigated, is the ‘tachyonic boundaries’, when the pairs of matter and Liouville boundary values themselves satisfy the mass-shell condition required by BRST invariance. Such correlators satisfy the pair of equations (2.8) and (2.12), with correlated signs $\delta, \bar{\delta}$, preserving the chosen (chirality) type of the relation. For instance, choosing degenerate $c < 1$ values for the matter boundaries hence implies that the spectrum of the Liouville boundary labels $\{\sigma_i\}$ is the same as that of the representations, i.e. (4.2) up to reflections. These solutions of equations (2.8) and (2.12) could be rather interpreted as the ‘string q-6j symbols’.

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Appendix A. Fundamental OPE coefficients data

The fusing matrix elements and the boundary OPE constants in (2.3) and (2.5), containing a fundamental $c > 25$ or $c < 1$ Virasoro representation, are known Coulomb gas constants, e.g.

$$F_{s,t}^L = F_{\beta_1 - s\frac{b}{2}, \beta_2 - t\frac{b}{2}}^L \begin{bmatrix} \beta_2 & -\frac{b}{2} \\ \beta_3 & \beta_1 \end{bmatrix} = \frac{\Gamma(tb(Q - 2\beta_2))\Gamma(1 - sb(Q - 2\beta_1))}{\Gamma(\frac{1-s}{2} + tb(\beta_3 - \beta_2 + st\beta_1 - st\frac{b}{2}))\Gamma(\frac{t+1}{2} - tb(\beta_2 + \beta_3 - st\beta_1 - \frac{b}{2}) - \frac{s-t}{2}bQ)}. \quad (\text{A.1})$$

The dual fusing matrix elements $\tilde{F}_{s,t}^L$ are obtained with $b \rightarrow 1/b$. All these expressions should be considered to be furthermore restricted by the fusion rules; for example, $F_{++}^L = 1$ if $\beta_1 = 0$ since the fusion rule leads to $\beta_3 = \beta_2 - b/2$ or if $\beta_3 = Q$ so that $\beta_1 - b/2 = Q - \beta_2$. The expression for the matter fundamental fusing matrix elements is obtained from (A.1) by analytic continuation $b^2 \rightarrow -b^2$ and $b\beta_i \rightarrow b e_i$ (so that $b(\beta_1 - tb/2) \rightarrow b(e + tb/2)$):

$$F_{s,t}^M := F_{e_1 - s\frac{b}{2}, e_2 - t\frac{b}{2}}^M \begin{bmatrix} e_2 & \frac{b}{2} \\ e_3 & e_1 \end{bmatrix} = \frac{\Gamma(tb(2e_2 - e_0))\Gamma(1 + sb(e_0 - 2e_1))}{\Gamma(\frac{1+s}{2} - tb(e_3 - e_2 + ste_1 + st\frac{b}{2}))\Gamma(\frac{1-t}{2} + tb(e_2 + e_3 - ste_1 + \frac{b}{2}) + \frac{s-t}{2}be_0)}. \quad (\text{A.2})$$

The dual $\tilde{F}_{s,t}^M$ is recovered from $F_{-s,-t}^M$ by the change $b \rightarrow -1/b$. For the choice of the chiralities of the three fields as in (2.6), one has $F_{s,t}^M = F_{-s,t}^L$, $\tilde{F}_{s,t}^M = \tilde{F}_{-s,t}^L$ which implies (2.7). Furthermore, one needs the particular fundamental OPE coefficient in (2.3) and (2.5). In the Liouville case, it is given by [8]

$$C_{\sigma_3 \pm \frac{b}{2}, \beta_1 \pm \frac{b}{2}}^L \begin{bmatrix} -\frac{b}{2} & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} = -\frac{b^2 \sqrt{\lambda_L} \Gamma(1 - 2b\beta_1)}{\Gamma(1 + (Q - 2\beta_1)b)} c_{(\mp)}^L(\sigma_3, \beta_1, \sigma_1) \quad (\text{A.3})$$

with the last constant being written down in (2.9). The corresponding matter factor (obtained also as analytic continuation of (2.9)) reads as

$$C_{\bar{\sigma}_3 \mp \frac{b}{2}, e_1 \mp \frac{b}{2}}^M \begin{bmatrix} \frac{b}{2} & e_1 \\ \bar{\sigma}_3 & \bar{\sigma}_1 \end{bmatrix} = b^2 \frac{\sqrt{\lambda_M} \Gamma(1 - 2be_1)}{\Gamma(1 + (e_0 - 2e_1)b)} c_{(\pm)}^M(\bar{\sigma}_3, e_1, \bar{\sigma}_1) \quad (\text{A.4})$$

with the explicit expression given in (2.10). Combining (A.3), (2.9) and (A.4), (2.10), the product of the coefficients in the lhs of the $t = t' = -1$ identities in (2.3), (2.5) reads as

$$C_{\bar{\sigma}_3, e_2 - \frac{b}{2}}^M \begin{bmatrix} e_2 & \frac{b}{2} \\ \bar{\sigma}_4 & \bar{\sigma}_2 \end{bmatrix} C_{\sigma_3, \beta_2 + \frac{b}{2}}^L \begin{bmatrix} \beta_2 & -\frac{b}{2} \\ \sigma_4 & \sigma_2 \end{bmatrix} = C_{\bar{\sigma}_3 = \bar{\sigma}_2 + \delta\frac{b}{2}, e_2 - \frac{b}{2}}^M \begin{bmatrix} \frac{b}{2} & e_2 \\ \bar{\sigma}_2 & \bar{\sigma}_4 \end{bmatrix} C_{\sigma_3 = \sigma_2 - \delta\frac{b}{2}, \beta_2 + \frac{b}{2}}^L \begin{bmatrix} -\frac{b}{2} & \beta_2 \\ \sigma_2 & \sigma_4 \end{bmatrix} = -\sqrt{\lambda_L \lambda_M} \frac{\Gamma(\frac{1}{b}(Q - 2\beta_2 - b))}{\Gamma(\frac{1}{b}(Q - 2\beta_2 + b))} c_{(-\delta)}^M(\bar{\sigma}_2, e_2, \bar{\sigma}_4) c_{(\delta)}^L(\sigma_2, \beta_2, \sigma_4). \quad (\text{A.5})$$

The Γ 's in (A.5) are eliminated by the leg factor normalization (2.1) and by collecting everything, we obtain relation (2.8) for the normalized constants \hat{C} :

$$\begin{aligned} & \frac{1}{\Gamma(b(2\beta_3 - Q))} \hat{C}_{\sigma_3, \beta_3} \begin{bmatrix} \beta_2 & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} \\ &= \Gamma\left(\frac{1}{b}(Q - 2\beta_2)\right) \Gamma(b(Q - 2\beta_1)) C_{\sigma_3, \beta_3}^L \begin{bmatrix} \beta_2 & \beta_1 \\ \sigma_4 & \sigma_1 \end{bmatrix} C_{\bar{\sigma}_3, e_3}^M \begin{bmatrix} e_2 & e_1 \\ \bar{\sigma}_4 & \bar{\sigma}_1 \end{bmatrix} \end{aligned}$$

Appendix B. Examples

Example 1. $e_{123} = e_0 + b = 1/b$.

The matter formula (3.4) reads (set $\lambda_M = 1$) as

$$\begin{aligned} C_{\bar{\sigma}_2, e_0 - e_3}^M \begin{bmatrix} e_2 & e_1 \\ \bar{\sigma}_3 & \bar{\sigma}_1 \end{bmatrix} &= -\frac{\Pi_M(e_3, e_2, e_1)}{2 \sin \pi b^2 2 \sin \pi 2e_1 b} (c_{(-)}^M(\bar{\sigma}_3, e_3 - b/2, \bar{\sigma}_1) + c_{(+)}^M(\bar{\sigma}_3, e_2 - b/2, \bar{\sigma}_2)) \\ &= \frac{b^2 \Gamma(b^2) \prod_i \Gamma(1 - 2e_i b)}{(2\pi)^2} 2(\sin \pi 2e_2 b c_1^M + \sin \pi 2e_3 b c_2^M + \sin \pi 2e_1 b c_3^M), \end{aligned} \tag{B.1}$$

where $c_i^M = 2 \cos \pi b(b + 2\bar{\sigma}_i)$. The cyclic symmetry of the three-point function is explicit. As a particular example, one recovers from (3.4) the OPE constant (A.4), leading to (2.10).

We shall use (B.1) for the particular choice of three degenerate matter fields:

$$\beta_1 = \beta_2 = \beta_3 = 2b \rightarrow e_1 = e_3 = b = e_0 - e_2 \tag{B.2}$$

or any other choice of two (+) and one (-) chiralities. For our example, $k = 2$ in (4.9) is even and the polynomial (4.9) in the variables c_2, c_1

$$B(\sigma_2, \sigma_1)^{(2;0)} = (c_2 - c_1) g_- \left(\sigma_2, -\frac{b}{2}, \sigma_1 \right) g_- \left(\sigma_2, \frac{3b}{2}, \sigma_1 \right) = (c_2 - c_1) P(c_2, c_1) \tag{B.3}$$

is antisymmetric. The polynomial (4.11) is symmetric in σ_1, σ_3 and is proportional to

$$P_2 := g_- \left(\sigma_2, -\frac{b}{2}, \sigma_1 \right) + g_- \left(\sigma_2, \frac{3b}{2}, \sigma_3 \right) = -\sum_{i=1}^3 c_i + c_2(1 + 2 \cos \pi 2b^2). \tag{B.4}$$

Then the Liouville factor (4.12) reads as

$$\begin{aligned} L C_{2b, 2b, 2b}^{\sigma_3, \sigma_2, \sigma_1} &= \frac{S_b(3b) S_b(2b)}{S_b^2(b)} \frac{\lambda_L^{\frac{Q-6b}{2b}} \Pi_L(2b, 2b, 2b)}{B(\sigma_1, \sigma_2)^{(2;0)} B(\sigma_2, \sigma_3)^{(2;0)} B(\sigma_3, \sigma_1)^{(2;0)}} \\ &\times \det \begin{pmatrix} \tilde{c}_3 X_3 & \tilde{c}_2 X_2 & \tilde{c}_1 X_1 \\ c_3 & c_2 & c_1 \\ 1 & 1 & 1 \end{pmatrix} \end{aligned} \tag{B.5}$$

with $X_3 = X_3(c_1, c_2, c_3) := P_3 P(c_1, c_2)$. Combining (B.5) with (B.1) and the full prefactor from (4.7), one obtains the tachyon correlator in this example. Note that for the choice of the chiralities $-\epsilon_1 = 1 = \epsilon_2 = \epsilon_3$, the matter correlator (B.1) is indeed proportional to the polynomial P_3 in (B.4), since all c_k^M are identified with c_k . Similarly, the choice of the negative chirality as $\epsilon_2 = -1$ or $\epsilon_3 = -1$ leads to the polynomial P_1 or P_2 respectively.

Example 2. $e_{123} = e_0 - \frac{1}{b} = -b$.

The matter formula (3.4) reads as

$$\begin{aligned}
 C_{\bar{\sigma}_2, e_0 - e_3}^M \begin{bmatrix} e_2 & e_1 \\ \bar{\sigma}_3 & \bar{\sigma}_1 \end{bmatrix} &= \frac{\Pi_M(e_3, e_2, e_1)}{2 \sin \pi / b^2 2 \sin \pi 2e_1 / b} \left(\tilde{c}_{(+)}^M \left(\bar{\sigma}_3, e_3 + \frac{1}{2b}, \bar{\sigma}_1 \right) + \tilde{c}_{(-)}^M \left(\bar{\sigma}_3, e_2 + \frac{1}{2b}, \bar{\sigma}_2 \right) \right) \\
 &= \frac{1}{b^2} \frac{\Gamma(\frac{1}{b^2}) \prod_i \Gamma(1 + \frac{2e_i}{b})}{(2\pi)^2} 2 \left(\sin \pi \left(-\frac{2e_2}{b} \right) \tilde{c}_1^M + \sin \pi \left(-\frac{2e_3}{b} \right) \tilde{c}_2^M \right. \\
 &\quad \left. + \sin \pi \left(-\frac{2e_1}{b} \right) \tilde{c}_3^M \right), \tag{B.6}
 \end{aligned}$$

where $\tilde{c}_i^M = 2 \cos \pi \frac{1}{b} (\frac{1}{b} - 2\bar{\sigma}_i)$. Comparing with (B.1), one observes that the symmetry $b \rightarrow -1/b$ of the correlator is indeed confirmed. The matter correlator (B.6) can be used, e.g., to compute the tachyon three-point function with

$$\beta_3 = \beta_2 = \beta_1 = b - 1/b \quad \Rightarrow \quad e_1 = e_3 = -1/b = e_0 - e_2.$$

The Liouville three-point function in this case has been given in [19] and it is cast in a form similar to (B.5):

$${}^L C_{b-\frac{1}{b}, b-\frac{1}{b}, b-\frac{1}{b}}^{\sigma_3, \sigma_2, \sigma_1} = \frac{S_b(\frac{1}{b}) S_b(\frac{3}{b}) \lambda_L^{\frac{Q-3e_0}{2b}} \Pi_L(b - \frac{1}{b}, b - \frac{1}{b}, b - \frac{1}{b})}{S_b(\frac{2}{b}) S_b(\frac{5}{b}) (c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} \det \begin{pmatrix} c_3 \tilde{X}_3 & c_2 \tilde{X}_2 & c_1 \tilde{X}_1 \\ \tilde{c}_3 & \tilde{c}_2 & \tilde{c}_1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{B.7}$$

where \tilde{X}_i is the dual ($b \rightarrow 1/b$) of the polynomial X_i in (B.5). The duality $b \rightarrow 1/b$ transformation of (B.7), so that $\beta_i = b - 1/b \rightarrow 1/b - b$, gives a new correlator, which is obtained alternatively from (B.5) by reflecting all three boundary fields $\beta_i = 2b \rightarrow Q - 2b = 1/b - b$ with the corresponding two-point reflection amplitudes.

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